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THE UNIVERSITY OF ROCHESTER
THE INSTITUTE OF OPTICS

ROCHESTER, NEW YORK

RESEARCH ON FUNDAMENTALS OF
GEOMETRICAL OPTICS

Robert E. Hopkins

FINAL REPORT

October 31, 1962

Contract AF-49(638)-668

Project No. 976

Task No. 37650

**The Institute of Optics
The University of Rochester**

**Final Report
October 31, 1962**

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AFOBR

Summary

During the course of this contract an extensive series of programs have been prepared for an IBM 7070 data processing machine. This set of programs includes calculations of the following type.

1. Lens read in
2. Third and fifth order calculation
3. Automatic correction of third and fifth order calculations
4. Skew ray tracing modes
5. Radial energy distribution
6. Frequency response
7. Library routine

The ORDEALS Optics Programs

A set of programs have been prepared to carry out many of the requirements for automatic lens design on an IBM 7070. This set of programs is called ORDEALS.

The ORDEALS programs are described in the following writeups which have been prepared for general circulation. These notes were prepared for a Rochester Summer School where the programs were described and demonstrated. The programs written up are:

1. Optics Programs for the IBM 7070 Computer

This writeup is available in limited supply. It is also being constantly improved and added to. (Included with this report as Appendix A)

II. A general Linearization Method for Automatic Lens Correction.

This paper has not yet been presented for publication because all the latest features in this program have not been checked out. (Included with this report as Appendix B)

III. Introduction to the Geometric Optical Frequency Response

This report describes the theory justifying the use of Geometrical frequency response. (Included with this report as Appendix C)

Future Work

This ORDEAL program is now being put into subroutines in Fortran language suitable for a 7090. The new program will use automatic correction based on ray deviations rather than using aberration theory. By tracing 13 rays, it is possible to evaluate the optical system up to the 7th order. After experience with this method has been evaluated the decision will be made as to whether to add the third and fifth order aberration theory. The latest method will optimize using geometrical ray aberration. At a much later stage it will become necessary to incorporate wave front aberration correction.

Publications

During this period we have published the following paper in the Journal of the Optical Society.

Creative Thinking and Computing Machines in Optical Design. Robert E. Hopkins and Gordon Spencer. J. Opt. Soc. Am. Vol. 52, No. 2, 172-176, February 1962. This paper is included in this report.

The ORDEALS program writeup and tape units have been made available to three optical companies.

1. Texas Instruments
2. F. M. A.
3. Pacific Optical Company

The following companies have used the program at Rochester.

1. Pacific Optical Co.
2. Bausch and Lomb
3. Tropel Inc.
4. Eastman Kodak
5. Itek Corporation

Personnel

This contract has supported the graduate work of Gordon Spencer. He is now writing a PhD thesis on this problem of automatic correction.

The contract has also partially supported Mathew Rimmer. Mr. Rimmer is now completing a Masters thesis on Fifth Order Aberration Theory.

Mrs. Alice Sinclair has been used to write programs. Her main contribution has been in preparing a Library routine. This feature allows us to store for ready access all optical design solutions.

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Appendix A

OPTICS PROGRAMS FOR THE I.B.M. 7070 COMPUTER

May 10, 1962.

A set of design and evaluation programs called Ordeals have been written for the I.B.M. 7070 machine. The following write-up is a brief description of the program input and output features. The program is being constantly worked on, so this description is subject to alterations. There has been no attempt in this write-up to provide a complete description of the program, but it is available in autocoder language.

There are 238 pages of program. So far there are 75 pages of flow charts.

The Ordeals programs are written for an I.B.M. 7070 machine which contains the following basic units.

- 1) Three, channel one, tape units
- 2) Three, channel two, tape units
- 3) A card reader unit
- 4) A card punch unit
- 5) A 1401 printer unit
- 6) A 10K storage 7070.

The programs available are:

- 1) Automatic first, third and fifth order aberrations plus seventh order spherical aberration,
- 2) A ray trace program,
- 3) Spot diagram, radial energy distribution and pseudo geometrical frequency response.

The program can accommodate centered rotationally symmetrical surfaces of the form

$$z = \frac{cs^2}{1 + \sqrt{1 - (K+1)c^2s^2}} + (e)s^4 + (f)s^6 + (g)s^8 + (h)s^{10}, \quad (1)$$

where $s = \sqrt{x^2 + y^2}.$

Decentering and tilt plus non-rotationally symmetrical surfaces are being added to the program. A write-up of Spryte (Special Purpose RAY TRACE) is included in Appendix #1.

The 7070 Ordeals program has been planned and written by Mr. Mathew Rimmer and Mr. William Hennessy. Mr. Rimmer wrote the aberration analysis, and Mr. Hennessy planned and wrote most of the other features in the program. Mr. Gordon Spencer has worked out the details of most of the computation techniques, and formulated the automatic correction method used. (See Appendix 2.) Dr. Robert Hopkins has mostly contributed by insisting that the program be easy to use and require a minimum of input data.

A. The Ordeals Programs.

All of the 7070 Ordeals programs are located on tape 10. The major programs are called into core storage for execution with call cards. The call cards are simply cards with the name of the program punched on them. There are three segments of programs. One set, called the I segment, remains in core storage once it has been called in. The programs in segments one and two have to share space in core storage. If one calls programs from segment two, the programs from segment one which are in core are wiped out. The programs with their call names and segments are listed below:

Segment I.

LENIN	Lens read in.
LEO	Lens punch out.
LOAD	Load constants.

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Segment 1.

AXPAN	Axial fan.
FUFTR	Full field trace.
INDVR	Individual ray.
SAM	Spot diagram.
SHIFT	Focus shift.
RED	Radial energy distribution.

Segment 2.

FORD	Fifth order
COORD	Fifth order correct.
AUTO	Automatic correction.

Segment 1 contains programs used in connection with ray tracing. Segment 2 contains programs connected to aberration calculation and correction. Segment I contains programs common to ray tracing and aberration analysis.

B. BASIC FEATURES OF CALL AND DATA CARDS.

Input cards are always punched with alphabetic information in the first field. Either alphabetic or numeric information may be punched in the five remaining fields. Fields are separated on the card by one or more blank columns. The first field may be punched anywhere on the card, but usually it is started in the first column. A field may contain a maximum of ten alphabetic characters or twenty numeric characters.

In most cases, numerical information is punched in floating point form, with the sign first, the exponent next and the number last. For example:

511000000 represents the number + 1.0 and
- 491000000 represents the number - 0.01.

The floating point exponent refers to 50 as a base, and exponents may go from 99 to 01. It is not necessary to punch the insignificant zeros in a number; hence the number -0.00400000 could be punched -484, and the above numbers as 511 and -491.

If all the parameters on a card are zero, the alphabetic field is the only one which must be punched. If the first parameter is zero, but the second is not, then it is necessary to punch a single zero in order to keep the proper order of the words. For example, an aspheric constants card may have four fields plus the alphabetic field. If the first, second and fourth constants are zero and the third constant is -0.000026 , it may be punched as follows:

ASPH 0 0 -4626.

C. MAKE UP OF THE LENS DECK.

Since there are several options available for setting up a lens deck, a general outline is presented here.

1. LENIN. This call card calls the lens read-in program, reads the lens data and executes a paraxial trace.
2. Comment cards: Cards may be inserted to identify lens, give date, etc.
3. Index selection card: This card defines the wave lengths to be used in the third and fifth order aberration computations. Five wave lengths are specified: mid-wave length, two for primary color aberrations and two for secondary color aberrations.
4. First curvature card: defines the curvature of the object surface.

Appendix A

5. First thickness card: defines the axial separation between the object surface and the first surface of the lens system.
6. Index of Refraction card: This card defines five indices of refraction (usually, A, C, d, F, g) for the medium following the surface. Any five wave lengths may be used, but they are always referred to as A, C, D, F and G (in that order) in the program.
7. Curvature, thickness and index cards for the remaining surfaces. Thickness and index cards refer to the space following the surface.
8. A final curvature card for the final image surface.
9. The paraxial ray data card.
10. Call and data cards for other programs.

D.

INPUT DATA

Details of Input and Call Cards.

1. LENIN: This is a call card for the lens read-in program. LENIN may be punched in any position in the card. All the letters must be together, however, with no spaces between them.
2. Comment cards: Following the LENIN card, one may add as many comment cards as desired. A comment card must start with an asterisk in word field one. Comments in numeric or alphabetic form may be punched according to the following rules:
 - a. Words should not exceed ten digits.
 - b. Do not extend comments beyond column 59.

- c. There must be a space between the asterisk and the first character.

A card with an asterisk may be used as a spacing card.

Example: The comments, "John Smith, Telescope Objective, December 8, 1962", could be punched in the following four cards:

```
*  
* JOHN SMITH TELESCOPE OBJECTIVE  
* DEC 8 1962  
*
```

3. Index of Refraction selection card:

Field one is punched with the alpha word NRSEL. Field two specifies the wave lengths used in the calculations, the first letter designating the major color. This wave length will be used for the paraxial trace, third and fifth order aberrations. The next two letters refer to the primary chromatic aberrations; the last two letters refer to the secondary chromatic aberrations. Any permutation of ACDFG with repetitions, making a total of five letters, is permitted. Note that "FC" and "FD" will give chromatic aberrations with the usual sign conventions, while "CF" and "DF" will give aberrations with reversed signs.

Example: To do third order and fifth order calculations in D light, primary color between F and C and secondary color between F and D, punch the NRSEL card as follows:

```
NRSEL      DFCFD
```

4. Curvature Cards:

There are the following types of curvature cards which may be

Appendix A

used in the program:

<u>Card Type</u>	<u>Code for Field 1</u>
a) Regular curvature	CV
b) Regular curvature, stop surface	CVS
c) Regular curvature, image surface	CVI
d) Angle solve on the axial ray	UA
e) Angle solve on the chief ray	UC
f) Radius of curvature	RD
g) Previous curvature pick up	PC
h) Previous surface pick up	PS
i) Previous radius pick up	PR

The S may be added to the code for any curvature card in order to make the surface the stop surface. For example, PCS, UAS are allowed codes for curvature cards. An I may be added to any curvature code in order to make the surface the image surface, e.g., PCI.

The first curvature card in a lens system is always the object curvature card. Usually it has a curvature of zero, so it is punched CV. The zero is not needed.

Previous curvature pick up: Often it is necessary or desirable to make a curvature identical with, or have a constant difference from, a previous curvature. This is done with the code PC. In field two the surface number of the surface to be picked up is inserted in fixed point. The curvature may be picked up with a reverse in sign by punching a minus sign in front of the surface number. A constant difference ($\pm C$) may be added (or subtracted) to the picked up curvature by inserting a floating point number in field three. The conic constant K will always be picked up, and its sign will not be changed.

Example: Suppose that we want the 15th surface to pick up the curvature on the 10th surface with a reverse in sign and a curvature difference of -0.01. Punch the 15th curvature card as follows:

PC -10 -491.

If the code PR is used, it picks up the previous radius and it adds a ΔR .

If the code is PS, it picks up all the surface constants of the previous surface, including the constant K (see Equation 1), and the aspheric constants if there are any.

Conic sections: Any floating point numbers inserted in the third word field of a CV card are interpreted as the conic section constant K. The following table relates the conic section K to the various types of conic sections available:

<u>Type of Conic Section</u>	<u>Constant K</u>
Paraboloid	$K = -1$
Hyperboloid	$K < -1$
Ellipsoid revolved about the major axis	$-1 < K < 0$
Ellipsoid revolved about the minor axis	$K > 0$

The conic section $K = -E^2$ where E is the eccentricity.

Example: A paraboloid could be described as CV -491 -511.

Aspheric Surfaces: General aspheric surfaces of the type given in equation 1 may be inserted at any time by adding a card directly after a curvature card, with ASPH in field one. Fields two, three, four and five are reserved for the coefficients e, f, g and h.
(See equation 1.)

Appendix A

Example: An aspheric surface with deformation from a paraboloid would then have two cards as follows:

CV -491 -511

ASPH 0 0 -4626 -4524

5. Thickness cards:

There are several types of thickness cards, as indicated in the following table:

<u>Card Type</u>	<u>Code for Field 1</u>
a) Regular thickness	TH
b) YA solve. The computer computes the required thickness to make the y of the axial ray on the next surface have the prescribed YA.	YA
c) YC is the same for the chief ray.	YC
d) Previous thickness pick up. This is exactly analogous to the previous curvature code.	PT
e) Clear aperture solve.	CA

Clear aperture solve: This feature provides for appropriate axial separations between surfaces. There are two types of adjacent surfaces. They are called closed if they edge contact when brought together. If they contact at the center, they are called open. If the code CA is used on the j surface, the program checks to see what the $(j + 1)$ surface is. Next it checks to see whether the material between the two surfaces is glass or air. Finally it computes the thickness between the j and the $(j + 1)$ surfaces according to the following schedule:

<u>Condition</u>	<u>Air</u>	<u>Glass</u>
) (Open	$TH_j = a$ a nominally is 0.01	$TH_j = b \text{ CA.}$ b nominally is 0.2
() Closed	$TH_j = Z_j - Z_j + 1$	$TH = Z_j - Z_j + 1 + c$ c nominally is 0.1

In no case is the glass thickness made less than 0.2 CA. The second field of the CA card contains the numerical value of the radius of the clear aperture required.

For example: A clear aperture radius of 1.2 would be punched:

CA 5112.

The sags Z_j and $Z_j + 1$ are computed with the following approximate formula:

$$Z = CS^2 \left(1 - \frac{C^2 S^2 (K + 1)}{4} \right) / (2 - C^2 S^2 (K + 1))$$

One should remember that the edge thicknesses (c) of the lenses are determined by the fixed constant 0.1. This constant may be altered at any time, as well as the constants a and b, by inserting LOAD cards after the FORD card. See the section (G) on altering program constants.

Object Distance: In this program all object distances TH_0 are considered to be finite. To make an infinite object distance, one merely needs to use a large number for TH_0 . For example, it is convenient to use a floating point number such as 601, or 60F, where F is the focal length of the system. This is useful because then the object height \bar{Y}_0 has the same value as the image height, with merely a change of the exponent. For example, suppose a lens has a focal length of 24" and it is to cover a half image field of 6". Then if TH_0 is made 6024, \bar{Y}_0 is 596. There is one danger with

Appendix A

this if the first surface is plano or slightly concave towards the object. Then i_1 which is equal to $yc + u_1$ may become so small that, when it is raised to the 7th power, as is done in calculating the 7th order spherical aberration, it may underflow. If this happens, one may reduce TH_0 .

6. Index of Refraction Cards:

a) Refracting surface: Punch INDEX in word field one and five floating point indices in word fields two, three, four, five and six.

The five indices are referred to by the alphabetic letters A, C, D, F and G. One may insert any indices desired, but they must be referred to with these letters.

Caution!:

1) Punch an index other than zero for all five fields even though they may not be used.

2) The index card is a potential source of grave danger.

A misspunched number here can mean a false index of refraction and go unnoticed until the final design is made up. Be sure and check these indices! It is advisable to compute V numbers directly from the numbers on the card. It is also advisable to make up a set of prepunched cards of the commonly-used glasses.

b) Reflecting surfaces: If a surface is to be reflecting, simply punch REFL in word field one in place of INDEX. No other numbers are needed. The program automatically inserts negative index values for all the following surfaces until another REFL card is encountered. During this right-to-left travel, one must insert

thicknesses with negative values. This is one place where the negative previous thickness is useful.

7. Paraxial Ray Data Card:

A card with PXRAY in field one is used to provide the specification of the axial and the chief ray.

The items entered in fields two, three and four are:

\bar{y}_0 - the height of the object at the object surface. It is usually chosen to be equal to the maximum value to be used by the lens.

y_1 - the height of the axial ray at the first surface of the system. It is usually chosen to be the maximum height acceptable by the lens for axial rays.

\bar{y}_1 - the height of the chief ray on the first surface.

If a surface in the system is labelled as the stop surface, it is not necessary to insert a value of \bar{y}_1 . The program will assume a value of \bar{y}_1 and then trace the chief ray through to the stop surface. If the \bar{y} is not zero on the stop surface, the program automatically changes \bar{y}_1 to make it zero. If no surface is labelled stop surface and \bar{y}_1 is left zero, then the first surface becomes the stop surface.

8. FORD:

FORD is a call card which calls in the aberration correcting program and initiates the computation. Upon completion, the program automatically transfers to CORD.

9. CORD:

This program initiates the automatic correcting part of the program. The first thing it does is ask to read VARY cards, and finally CORR, ADD, HOLD AND HOLDT cards. These cards may be explained

Appendix A

as follows:

a) VARY cards.

VARY cards are punched with VARY in field one, a fixed point surface number in field two, the curvature, conic constant k, aspheric constants - or thickness to be varied in field three, and a fixed point weight in field four. The weights vary from zero to nine.

Example: To vary the curvature on the 6th surface with a weight of two, punch a card as follows:

VARY 6 CV 2

It is possible to vary CV, CC*, TH, AD, AE, AF, AG, RD. If a VARY CV card is used, the curvature on the surface will be varied without regard to the curvature code used. For example, if the surface has a UA code and a VARY CV card is used, the UA will be varied.

b) CORR cards.

These cards specify the items for correction. CORR is punched in field one. Field two is the aberration to be corrected. Field three is the target value for the correction. The items which may be corrected are listed in part f of this section.

Example: If one wishes to correct the spherical aberration SA5 to zero, the card can be punched

CORR SA5

c) ADD cards.

If a CORR card is changed to an ADD, then the aberration signified in word field two is added to the previous CORR card.

Never punch target values or weights on an ADD card.

* CC is the code for the conic section constant K.

Example: One may correct $SA3 + SA5 + SA7$ to zero by inserting the following three cards:

CORR SA3

ADD SA5

ADD SA7

One could correct a system to an over-all length of 10.0 by inserting the following set of cards.

CORR TH 1 521

ADD TH 2

ADD TH 3

etc. to

ADD TH K-1

d) HOLDT card.

If the CORR is replaced by HOLDT in word field one, the program tries to minimize the difference between the target value and present value. Field two contains the aberration, field three the target value and field four a fixed point weight which varies from zero to nine.

Example: If one wants to minimize the difference between $SA3 + SA5 + SA7$ and zero with a weight of 2, one can punch the following three cards:

HOLDT SA3 0 2

ADD SA5

ADD SA7

The HOLDT cards are used for aberrations which may not lend themselves for complete correction to a target value. One also may use HOLDT cards to correct aberrations after all the variables are used up. One may insert more HOLDT cards than variables.

Appendix A

e) HOLD cards.

If HOLDT is replaced by HOLD in field one, it indicates that the present value of the aberration should become the target value. Otherwise the card is exactly the same as the HOLDT card.

f) The following items may be corrected with CORR, HOLDT, AND or HOLD cards:

FOCAL	The focal length of the system.
TACH	Primary axial color.
TCH	Primary lateral color.
TACH2	Secondary axial color.
TCH2	Secondary lateral color.
SA3	Third order spherical aberration.
SA5	Fifth order spherical aberration.
SA7	Seventh order spherical aberration.
COMA3	Third order tangential coma.
COMA5	Fifth order linear coma.
LCOMA	Seventh order cubic coma.
TAS3	Third order tangential astigmatic blur.
TAS5	Fifth order tangential astigmatic blur.
DIST3	Third order distortion. ΔY_k
DIST5	Fifth order distortion. ΔY_k
TOBSA	Tangential oblique spherical aberration.
SOBSA	Sagittal oblique spherical aberration.
PTZ3	Third order Petzval blur.
PTZ5	Fifth order Petz blur.
SAS3	Third order Sagittal astigmatic blur.
SAS5	Fifth order Sagittal astigmatic blur.

If only third order aberrations are asked for, the program does not compute all the fifth order aberrations during the automatic correcting phase.

g) It is possible to correct some items on individual surfaces. In this case, the CORR, HOLD, ADD, or HOLDT are punched as follows:

Field 1.	CORR, ADD, HOLD or HOLDT.
Field 2.	CODE OF SURFACE ITEM.
Field 3.	Surface number.
Field 4.	Target value.

The codes for the individual surface items are:

PI	Angle of incidence of axial ray.
• PIC	Angle of incidence of chief ray.
PU	Angle u of axial ray.
PUC	Angle \bar{u} of chief ray.
TH	Thickness.

CORD will transfer to read in correction data. After CORD, one should have VARY, CORR, ADD, HOLD, or HOLDT cards. These cards are followed by an AUTO card described in the next section. FORD automatically transfers to CORD upon completion of the fifth order program. Thus it is not necessary to follow a FORD card with a CORD card. If FORD is followed by a call card for another program instead of a VARY, CORR, ADD, HOLDT or HOLD, the new program is called and control is transferred to it. CORD may be used to correct a lens in steps. For example, after FORD one might insert VARY cards, and CORR cards to correct axial and lateral color and Petsval. These must be followed by an AUTO card. Then a CORD card will permit reading in monochromatic correction data and an AUTO card, which will begin correction from the already color-corrected system.

Appendix A

10. AUTO calls in the automatic correction program and transfers to initiate the automatic correction procedure. It must be used to initiate this feature.

The automatic correction method is briefly the following:

The c, t table will systematically be varied one parameter at a time. Depending on what the designer asks the machine to correct, it will compute third and fifth order aberrations. With this routine it is possible to compute finite difference ratios.

After all the required difference ratios are computed, they are loaded into the equation solving routine.

The equation solving routine seeks to minimize a function

$$\phi = \sum_{i=1}^M \left(\sum_{j=1}^N a_{ij} x_j - d_i \right)^2$$

subject to the simultaneous solution of the set of equations,

$$\sum_{j=1}^N b_{lj} x_j = e_l, \quad l = 1, 2, \dots, L$$

In matrix notation this leads to the solution of the following set of $N + L$ equations with $N + L$ unknowns:

$$\begin{aligned} A^T A x + B^T \lambda &= A^T d, \\ Bx + 0\lambda &= e \end{aligned}$$

with

$$\begin{aligned} A &\equiv \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \dots & a_{MN} \end{pmatrix}, & B &\equiv \begin{pmatrix} b_{11} & \dots & b_{1N} \\ \vdots & & \vdots \\ b_{L1} & \dots & b_{LN} \end{pmatrix} \\ x &\equiv \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, & d &\equiv \begin{pmatrix} d_1 \\ \vdots \\ d_M \end{pmatrix}, & e &\equiv \begin{pmatrix} e_1 \\ \vdots \\ e_L \end{pmatrix}, & \text{and } \lambda &\equiv \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_L \end{pmatrix} \end{aligned}$$

These equations are used in the design program to solve the sets of non-linear equations encountered in lens design by assuming linearity over a small region.

The elements of the x matrix become $p_j - {}_0p_j$. The difference ratios are evaluated at ${}_0p_j$. p_j is the value of p_j at a predicted solution.

The elements of the d matrix become d_j , a set of distances from target values for the set of functions to be minimized.

The elements of the e matrix become e_j which are the distances from target values for the functions to be corrected exactly.

The λ matrix is a set of Lagrange multipliers.

This method is the general form for McCarthy's, Rosen's and Wynne's method.

A more detailed analysis of this procedure is described by Gordon Spencer in a paper presented to the Optical Society of America and included in Appendix 2.

11. Ray Trace Modes.

There are four modes of ray tracing available. The first three modes are called with the cards AXFAN, FUFTR, INDVR. The fourth mode is SAM, a ray tracing for spot diagrams. This will be described in Section 13. The first two call in programs which use the paraxial ray data and automatically trace through a group of rays. The last card calls for data specifying individual ray data. A description of these modes is as follows:

a) Axial Fan: This mode of trace is called by a card with the first field punched AXFAN. Immediately following this card, one must have a card punched with RADAT in field one and the wave length code for up to three wave lengths in field two.

Appendix A

Example: To trace axial fans in D light and C light, use the following two cards:

AXFAN

RADAT DC

To do more than three wave lengths, extra AXFAN and RADAT cards are needed.

The program traces three rays in the first wave length from the axial object point ($Y_0 = 0$) at the following values of X on the entrance pupil ($Y_{en} = 0$).

Ray 1 $X_{en} = 0.5 Y_{en}$

Ray 2 $X_{en} = 0.75 Y_{en}$

Ray 3 $X_{en} = 1.0 Y_{en}$

It then repeats the trace for the other wave lengths on the RADAT card.

b) Full Field Trace: This mode of ray trace is called with a card punched with FUFTR in field one. It is immediately followed with a RADAT card. This RADAT card is the same as the one used after AXFAN, but there are extra fields. In field three, a fixed point number specifies the number of off-axis image points. The fixed point number in field four specifies the number of meridional rays to be traced on each side of the chief ray.

Example: The following two cards are a common example of the FUFTR:

FUFTR

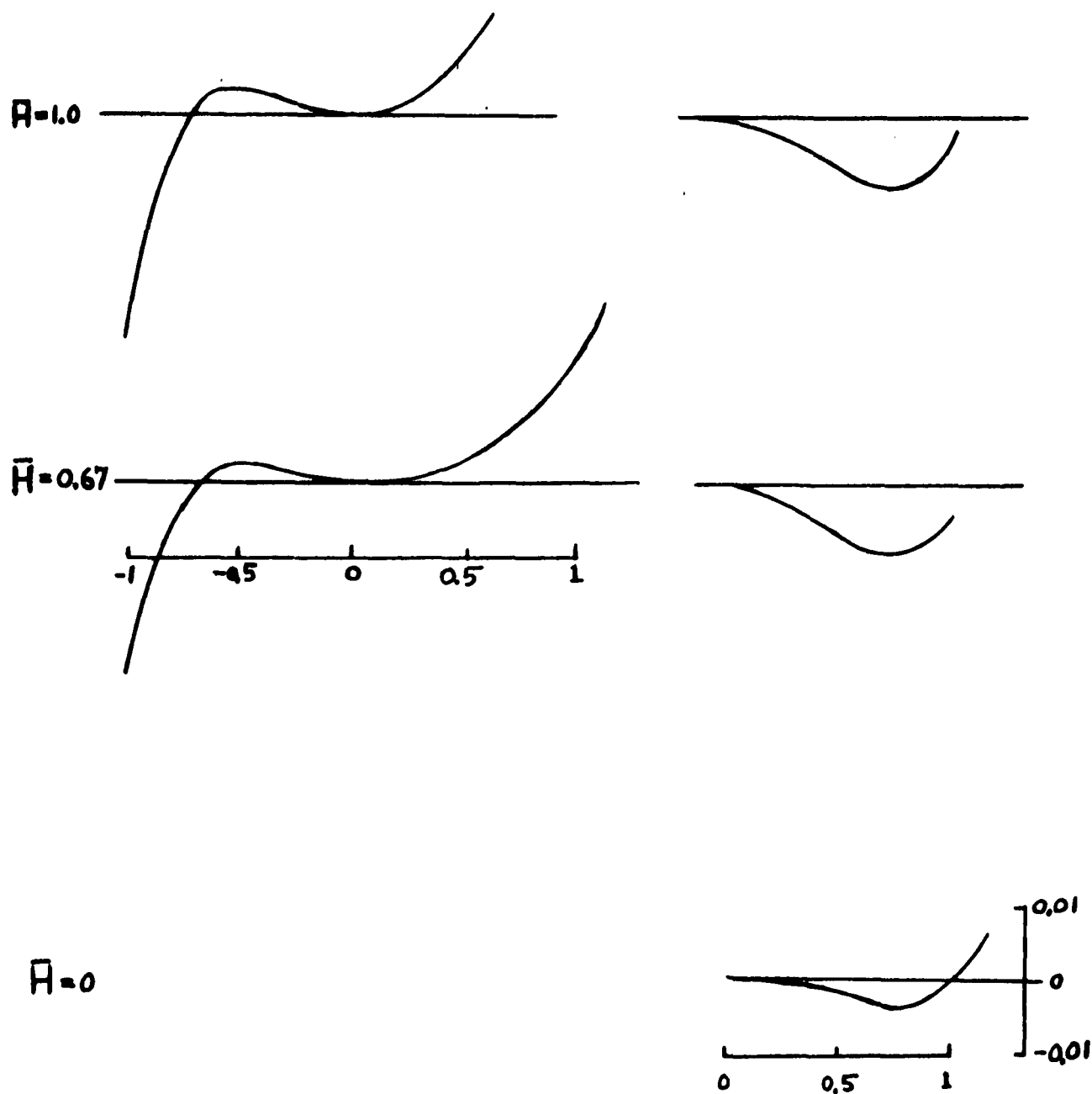
RADAT DFC 2 3

These cards call the following sequence of ray tracing:

- Rays 1-3 An axial fan in D light
- " 4-6 An axial fan in F light
- " 6-9 An axial fan in C light
- Ray 10 A chief ray in D light from an object point at $2/3 \bar{y}_0$.
- " 11 A chief ray in F light from an object point at $2/3 \bar{y}_0$.
- " 12 A chief ray in C light from an object point at $2/3 \bar{y}_0$.
- Rays 13-18 Rays from an object at $2/3 \bar{y}_0$ but confined to the meridional plane $X = 0$. The rays are evenly spaced above and below the chief ray. The two rays farthest from the chief ray pass through the top and bottom of the entrance pupil.
- " 19-21 Three skew rays from the object at $2/3 \bar{y}_0$ which pass through the entrance pupil with $Y_{en} = 0$. The rays are evenly spaced from the center of the pupil to the edge.
- " 22-24 Chief rays in D, F and C light from an object point at \bar{y}_0 .
- " 25-30 Meridional rays like rays 13-18 from an object point at \bar{y}_0 .
- " 31-33 Three skew rays like rays 19-21 from an object point at \bar{y}_0 .

Appendix A

The type of rays traced in this mode are illustrated in Fig. 1.



By punching PRINT in field five on RADAT card, one may obtain full surface print. (For description, see Section E.) With no PRINT one obtains partial print.

c) Individual Ray Trace: For individual ray trace, one must have the following arrangement of cards:

- 1) The first card is a transfer card with INDTR punched in field one.
- 2) The next card must be a fractional height card. FOBJH is inserted in field one; next the fractional object height in floating point is placed in field two. The rest of the card may be blank, in which case a minimum of printing will be done (no print), and color D will be selected for the chief ray. If a color other than D is desired (A C F G) one of these letters must be punched. If desired, PRINT may be punched first and then the color.

If the card read at this time is not a FOBJH card, the card will be ignored and reading will continue until a FOBJH card is read, or a transfer card to the next routine.

- 3) The next card must be ENPCO, an entrance pupil coordinate card. X on the entrance pupil is punched in field two and Y on the entrance pupil is punched in field three. The rest of this card may be blank or contain a color (A C D F G) or the word PRINT or the word NOPRT. If the card is blank, printing and color will be controlled by the FOBJH card, or if that was blank, no print and D will be used. The print and color codes are originally set to no and D respectively. These codes can be changed on either FOBJH or ENPCO cards and remain changed until a new change is read in. The color and print words can be entered in either order, or either may be blank, i.e., not put in at all.

Appendix A

12. Clear Apertures:

Provision is made in the program to stop rays which exceed a certain clear aperture radius. This is done to cut down on the amount of useless raytracing. The clear aperture radius may be arrived at in three different ways.

a) Automatic Assignment: If the designer makes no attempt to assign a clear aperture radius to a surface, the clear aperture is automatically computed from paraxial ray data. The clear aperture is computed from the following formula:

$$CA_j = |y_j| + |\bar{y}_j| + |0.2 y_j| + |0.2 \bar{y}_j|$$

In the early stages of design, the clear apertures are assigned in this manner.

b) By inserting a CA card in place of a TH card.

c) By inserting an OUTCA card. OUTCA cards are inserted just before the thickness cards. To provide for an outside clear aperture of 1.2, the card would be punched OUTCA 5112.

An OUTCA card should not be used on a surface with CA. The OUTCA card is used when the designer wishes to alter the clear apertures assigned automatically. There are cases where the paraxial ray data used in the automatic method is not accurate enough and rays are needlessly blocked. One example of this is the use of a field flattener. The formula assigns a value close to $CA = \bar{y}$ because y is very small. If there is any positive distortion, all the actual rays may exceed this value of \bar{y} . For this case it is advisable to insert an OUTCA card on the field flattener surfaces in order to overrule the automatic assignment of CA. OUTCA cards are used extensively in the later stages of

a design when the energy distribution is computed.

d) The program also provides for a card called INCA. This card blocks out all rays closer to the optical axis than a specified inside clear aperture radius. If a surface has both an OUTCA and INCA, they are inserted on two cards immediately preceding the thickness card. Their order does not matter.

The INCA card does not block out chief rays, since these rays are used as references.

13. Spot Diagram:

SAM is the call card for the spot diagram program.

a) SAM should be followed immediately by a card with WAVEW in field one. The fields two to six in this card are reserved for fixed point weights for the five wave lengths on the index of refraction cards. The weights are used to specify the number of rays to be traced. The weight times ten is the number of rays traced. For example: a WAVEW card punched 1 0 3 0 1 would trace 10 rays in A light, 30 rays in D light and 10 rays in G light. The spot diagram contains coordinates for all the wave lengths.

b) Immediately following WAVEW is a card with FOBJH in field one. Field two is reserved for the floating point fractional object height. This card, therefore, indicates the object height for the spot diagram. It is necessary to insert a FOBJH card for each object height.

c) If a focus shift is needed, a card with SHIFT is inserted after the FOBJH card. The second field on the SHIFT card should contain the focus shift as a floating point number.

Appendix A

d) The spot diagram coordinates may be listed by inserting a card with SPLIT in field one. All the preceding spot diagrams will then be listed.

The spot diagram program contains a subroutine that uses the paraxial ray data and the clear apertures of the lenses and computes the shape and area of the vignetted aperture on the entrance pupil. The rays that are traced for each color are evenly distributed over this area.

14. Radial Energy Distribution:

A call card labelled RED calls in the program to compute the radial or encircled energy distribution for all the preceding fractional object heights. This program computes the per cent radial energy distribution at five per cent intervals. The circles are all assumed to be centered around the chief ray in the major color as specified on the NRSEL card. It is possible to shift the center of the circles by inserting a card with CD in field one, followed by the floating point displacement in fields two and three for X and Y center displacement respectively. CD stands for center displacement. One may insert as many center displacement cards as needed.

15. FREQUENCY RESPONSE

This program computes a geometrical frequency response from energy distribution data. The formula used is

$$T(\omega) = \frac{\sum \Delta E J_0(\omega r)}{E_{MAX}}$$

where E is the difference between two successive values of the energy, $E \cdot r = (R)(RSC)$, where R is the arithmetic mean of the two E values, and $RSC = (\sec \theta)^{\frac{1}{2}}$, where θ is the obliquity angle,

$W = 2\pi N$ where N is the number of spatial cycles per unit length.
 R and N must be such that wr is dimensionless. Division by E_{\max}
normalizes the response function.

There are two entries to the program:

1) FREQT.

This call card will initialize computation of energy
distributions for each spot diagram computed thus far on
the present lens which will appear in the output.

- a) The obliquity factor and up to twenty values of N
are read in as follows:

RSC	+511
FRQ	+511
FRQ	+512
FRQ	513
FRQ	531
END	

- b) The output is two columns, one showing the frequency,
and the other the response at that frequency.

2) FREQC.

This card calls for energy distribution data to be
read from up to twenty cards with the code DIST in field
one, the energy in field two and the radius in field three:

DIST	495	485
DIST	501	489
DIST	511	492
END		

Appendix A

The program then transfers to read in the obliquity factor and the frequencies, as in 1A.

16. Lens Read Out:

A call card marked LEO is used to provide an updated lens deck. It may be inserted any place in a deck, provided it doesn't separate a call card and its required data cards.

E.

OUTPUT.

The output format for the 7070 programs are shown on pages 29-30. The first column is the line list number. The second column of three numbers indicates the type of printing. The 500 line is the input data obtained from the PXRAY card and the object distance TH_0 .

The lines 501 to 506 are the lines printed for each surface in the lens. The second column, which is a series of 01 values, is the surface number. In this example only the first surface is shown. The aberrations shown are the surface contributions.

The lines 507 to 510 are the total third and fifth order aberrations for the system.

The line 511 shows the printing during each change cycle. V_0 is the initial value of a parameter. V_1 is the new value. $V_1 - V_0$ is the change in the parameter.

The lines 200 to 253 are the lines indicating the printing in ray tracing. The last digit in the number indicates the wave length, according to the following rule:

A = 1
C = 2
D = 3
F = 4
G = 5

Line 223 shows the full print on individual surfaces.

The first line of printing after RED has the 150 printing code.

The line of printing is shown on page 30. The items printed in this line are:

1. The area of the vignetted aperture.
2. R_u is the radius of the aperture limiting the upper rays, as projected on the entrance pupil plane.
3. C_u is the center of this aperture as located in the entrance pupil.
4. R_l is the radius of the aperture limiting the lower rays, as projected on the entrance pupil plane.
5. C_l is the center of the lower aperture as located in the entrance pupil.
6. R_{en} is the radius of the entrance pupil.

Following this 150 line, the per cent energy table is printed.

Above the table four numbers are printed, and labelled. They are:

1. FOBJH fractional object height.
2. Focus shift.
3. CD_x center displacement in x direction.
4. CD_y center displacement in y direction.

The per cent energy table is made up of two columns, the per cent energy and the corresponding radius of spot.

Following the table, a single line of fixed point numbers gives the number of rays traced in each color and the total number of rays.

7070 OUTPUT FORMATLine
NumberCODE FOR
PRINTING
500INPUT DATAOBJ DIST OBJ INDEX OBJECT HT INVARIANT U_0 \bar{U}_0 SURFACE CONTRIBUTION DATA FOR FORD

(Each item is the jth surface contribution)

501 01	RADIUS	CONIC K	ASPH D	ASPH E	ASPH F	ASPH G
502 01	THICKNESS	INDEX N	TACH	TCH	TACH2	TCH2
503 01	SA3	COMA3	SAS3-PTZ3	DIST3	PTZ3	
504 01	SA5	COMA5	SAS5-PTZ5	DIST5	PTZ5	
505 01	SA7	LCOMA	TOBSA	SOBSA		
506 01	I	\bar{I}	U	\bar{U}	Y	\bar{Y}

ABERRATION TOTALS

507	EFL	IMAGE HT	TACH	TCH	TACH2	TCH2
508	SA3	COMA3	TAS3	DIST3	PTZ3	SAS3
509	SA5	COMA5	TAS5	DIST5	PTZ5	SAS5
510	SA7	LCOMA	TOBSA	SOBSA	EMP	ETP

CHANGES IN VARIABLES

511	V_0	V_1	$V_1 - V_0$	WEIGHT	VARIABLE
-----	-------	-------	-------------	--------	----------

RAY TRACE PARTIAL PRINT

200	\bar{Y}_0	Z_0			
213	X_1	Y_1	XEN	YEN	K_0/M_0
233	X_k	$Y_k - \bar{Y}_k$	XEX	YEX	K_k/M_k
243	\bar{Y}_k	\bar{Z}_k			L_k/M_k
253	Z_t	Z_s			

RAY TRACE FULL PRINT

SURFACE
NUMBER

223	O_j	X_j	Y_j	Z_j	K_j	L_j	M_j
-----	-------	-------	-------	-------	-------	-------	-------

PRINTING FOR RED

150	AREA	R_u	C_u	R_l	C_l	R_{em}
-----	------	-------	-------	-------	-------	----------

Appendix A

F. SAMPLE INPUT AND OUTPUT.

A sample Output from FORD is shown on pages 2 and 3 in Sample Design number three. The program first prints out the input data from LENIN to PIXRAY.

On Page 4 the Input from FORD to the final CORR DIST3 is shown. The remaining lines from 042 to 054 show the changes made in the first iteration and the total aberrations for the first iteration.

Page 15 shows the input cards AUTO, LEO and FUFTR. The remainder of page 15, pages 16, 17 and 18 show the ray tracing. The last machine printed line is the RADAT card for the ray tracing.

The makeup of the lens can always be checked by the input printing. The cards are all in sequence in the deck even though they appear to be spread apart in the printing.

The output for the energy distribution calculation is shown on pages 30 and 31 of the triplet example. The printing on this sheet is self-explanatory.

G. ALTERING FIXED CONSTANTS IN THE PROGRAM.

There are several constants that the designer may want to alter in the program. This may be done easily by introducing a card marked LOAD in field one. Field two should contain a four digit location number. Field three is reserved for the constant. The constant must be inserted with all ten digits.

The LOAD cards may be inserted directly after a FORD card. For example, see Page 12 of Sample 3.

The following constants are frequently changed:

<u>Use of Constant</u>	<u>Location</u>	<u>Programmed Value</u>
1) Maximum allowable change in a variable.	7116	+5020000000
2) Maximum number of passes.	5758	+0000000006
3) Tolerance on aberrations. The tolerance is a per cent of the target value plus a fixed increment. The per cent value is in The fixed increment is in	9589 9588	+5010000000 +4710000000
4) The constants a, b, c used in clear aperture solve	a b c	4278 4280 4279
		4910000000 5020000000 5010000000

H. OPERATION OF THE MACHINE. CARD ORIENTATION.

1. Set reader switch (1,2) to A, B.
2. Set punch switch (1,2) to B, B.
3. Mount Ordeals Tape on 10.
4. Mount Scratch Tapes on 11, 20, 22.
(20, 22 necessary only for SAM and RED.)
5. Initialize for Tape 10.
6. Press Start. End of File on Reader, Start on Punch.
7. Punch in time clock.
8. Press computer reset and start on typewriter console.
9. If typewriter types out CDERR
 - 1) Remove cards from stacker.
 - 2) Remove from hopper.
 - 3) Run out cards in machine.
 - 4) Add to cards in hopper and rerun by hitting start on typewriter.

Appendix A

10. In case of overflow or uncorrectable card error:

Depress Computer Reset and Start. This then bypasses the lens and goes on to the next lens. A comment RSTRT is printed out on print-out page.

11. Upon completion of problem, the console typewriter prints out the message.

Print tape 11 off line.

12. Use PEST output program for 1401.

13. Put output tape on tape unit 2.

14. Console Switch A up, all others down.

15. Put 8 $\frac{1}{2}$ x 11 paper in 1401.

I.

GENERAL COMMENTS.

1. The third and fifth order programs take approximately $\frac{1}{2}$ second per surface.
2. The ray trace takes 0.1 seconds per surface.
3. It takes approximately 10 seconds to do the matrix algebra required in a 40 x 40 matrix.
4. The size of the matrix is equal to the sum of the number of VARY cards plus the number of CORR cards.
5. The use of weights on the VARY cards is to change their relative influence with respect to other variables. As one increases the weight from 0 to 9 on a VARY card, it tends to decrease the change in the variable. It has been found that for lenses with a focal length around 10, it is not necessary to introduce weights (i.e., weights of zero) on the curvatures or thicknesses. In long focal length lenses, it is usually advisable to put some weight on the curvatures if it is desirable for them to influence the correction.

6. Weights may also be assigned to the HOLDT cards. A large number for the weight places a heavy emphasis on the minimization feature. If the weight is too large, it may make it impossible for the target values to be reached. On the other hand, if the weights are not high enough, the minimization may have little effect.
7. The problem of introducing weights is very difficult and our present program uses two methods for doing this. We are not satisfied with either method, so we are not including them in this write-up. Appendix three contains a discussion of the weighting problem, and describes how it is now being done.
8. It is advisable to limit the number of iterations to six. Usually something is amiss if this is not enough. In long systems it is good economy to cut this to four. This gives the designer a chance to see if everything appears satisfactory.
9. There is a mixed blessing written into the program. This is a maximum allowed change in variables. If any change exceeds this value, it is scaled to the maximum allowed change and all other changes are scaled by the same amount. This feature prevents many blow ups when solutions are far away from the target, but it also slows down the iteration. The value we have inserted in the program works well for lenses with a focal length around 10. It is much easier to insert this constant in hindsight than it is in advance.
10. There is a limit on the number of surfaces which may be used. The present limit is 30, including the object and image.

A General Linearization Method for Automatic Lens Correction

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The use of programmed computing machinery for automatic lens correction requires a definite prescription according to which a lens system may be adjudged: (a) acceptable or not acceptable, or (b) improved or not improved over a previous configuration. Judgments of the first kind may be made on the basis of whether or not a given set of equations are satisfied; judgments of the second kind, on the basis of whether or not the value of a "merit function" has been reduced. A typical lens design problem will involve both absolute requirements, to which a judgment of the first kind is appropriate, and relative requirements calling for a judgment of the second kind. This paper describes a linearization method designed to accommodate requirements of both types. Several previously described linearization procedures are shown to be included within the framework of the present method.

I. Introduction

The development of high speed programmed computing machinery has, during the past decade, prompted the investigation of various methods for automating, or partially automating, the lens design process. The problem is complicated, unfortunately, by the fact that the functions which measure system performance are non-linear in the system variables. One approach to the problem is to approximate the non-linear functions by linear ones and solve the linear problem. The iterative application of such a procedure can be expected to converge, under certain conditions, to the solution of the non-linear problem. This approach is adopted in the present paper.

In the method to be described, Lagrange multipliers are introduced as a means of achieving a sufficiently general framework to accommodate both absolute and relative system requirements. Absolute requirements are taken here to mean those requirements which must be met exactly in order that a system be considered acceptable. Relative requirements are those which may be lumped together, with various weighting factors, into a single "merit function" whose value is to be minimized. Experience indicates that it is desirable, if not necessary, to allow requirements of both types to be imposed in the course of designing a lens system.

It should be mentioned that the use of Lagrange multipliers is not new to automatic lens design. Feder¹, and Meiron and Lobenstein² have described modified steepest descent methods in which Lagrange multipliers are used to determine a direction of change for the system variables such that certain properties of the system remain unchanged while the value of a merit function is reduced. Steepest descent methods, however, possess serious deficiencies

from a practical standpoint, the chief among these being extremely slow convergence. There is some evidence to indicate that linearisation procedures, when they converge, are likely to converge much faster than descent methods.^{3, 4}

II. Statement of the Problem

One is given an optical system of J variables, p_j ($j = 1, \dots, J$).

These variables may be surface curvatures, axial separations of surfaces, refractive indices, or aspheric deformation coefficients. In terms of these variables, a number of functions, $f_k(p_1, \dots, p_J)$, may be defined which serve to measure various characteristics of the system, both with regard to its physical structure and its performance. The design problem is that of finding a simultaneous solution, or something approaching a simultaneous solution, of the set of equations

$$f_k(p_1, \dots, p_J) = s_k \quad (k = 1, \dots, K), \quad (1)$$

where the s_k are constants representing the desired values of the f_k .

Aside from such basic system properties as focal length, magnification, back focus, and total length, the choices which exist for the functions, f_k , are remarkably numerous. Among them one finds monochromatic and chromatic aberration coefficients, simple ray deviations referred to either the chief ray or Gaussian image points, mean spot diagram radii, weighted moments of spot diagram distributions, mean square wavefront deformations, and energy distribution and frequency response characteristics. Recent attempts to evolve single "figures of merit" for images of extended objects have produced such quantities as the relative structural content, fidelity defect, and correlation quality⁵. No attempt will be made here to assess the relative merits of these various choices. It should be pointed out, however, that considerations of physical significance (i.e., whether or not a meaningful indication is given

Appendix B

of the way in which a given object field will be imaged) must be supplemented by considerations of computational efficiency and linearity.

Now, frequently a simultaneous solution will not exist for the entire set of equations. If $K > J$, for example, the existence of a simultaneous solution is extremely unlikely. In such cases, the minimisation of

$$\Phi = \sum_{k=1}^K w_k^2 (f_k - s_k)^2, \quad (2)$$

where the w_k are weight factors, may be an acceptable alternative to an exact solution. If a simultaneous solution does in fact exist, it will correspond to the absolute minimum, $\Phi = 0$.

In general, there will be certain of the functions, f_k , for which it is essential that an exact solution be obtained, while for the remaining functions the minimisation of Φ will suffice. One thus divides the f_k into two groups,

$$g_m(p_1, \dots, p_J) \quad (m = 1, \dots, M)$$

and

$$h_n(p_1, \dots, p_J) \quad (n = 1, \dots, N < J),$$

and seeks to minimise

$$\Phi = \sum_{m=1}^M w_m^2 (g_m - s_m)^2 \quad (3)$$

subject to the simultaneous solution of the set of N equations,

$$h_n(p_1, \dots, p_J) = t_n \quad (n = 1, \dots, N < J), \quad (4)$$

where the t_n are the required values of the functions, h_n .

III. Linearisation of the Problem

As previously indicated, the functions, g_m and h_n , are generally non-linear. However, in the neighborhood of a given point, (p_1^i, \dots, p_J^i) , they may be represented by the constant plus first order terms of their Taylor expansions about that point. To this approximation, then,

$$g_m = g_m^i + \sum_{j=1}^J (\partial g_m / \partial p_j) (p_j - p_j^i) \quad (5)$$

and

$$h_n = h_n^i + \sum_{j=1}^J (\partial h_n / \partial p_j) (p_j - p_j^i), \quad (6)$$

where the derivatives are evaluated at the point (p_1^i, \dots, p_J^i) .

For notational convenience, the following definitions are now made:

$$\left. \begin{aligned} g_m - g_m^i &= d_m \\ h_n - h_n^i &= e_n \\ p_j - p_j^i &= q_j \\ \partial g_m / \partial p_j &= a_{mj} \\ \partial h_n / \partial p_j &= b_{nj} \end{aligned} \right\} \quad (7)$$

Using the linear approximations (5) and (6) for g_m and h_n , Eqs. (3) and (4) become, in terms of the definitions (7),

$$\Phi = \sum_{m=1}^M v_m^2 \left(\sum_{j=1}^J a_{mj} q_j - d_m \right)^2 \quad (3a)$$

and

$$\sum_{j=1}^J b_{nj} q_j = e_n \quad (n = 1, \dots, N < J). \quad (4a)$$

IV. The Method of Lagrange Multipliers

The problem of finding an extremum of a function subject to auxiliary constraints can be most profitably attacked by the method of Lagrange multipliers⁶. Although in the present context its application will be restricted to the minimisation of a quadratic function subject to linear constraints, the method is quite generally applicable.

Let the equations of constraint for a system of J variables be represented by

$$u_n(q_1, \dots, q_J) = e_n \quad (n = 1, \dots, N < J), \quad (8)$$

where the e_n are constants. Ordinarily such a set of equations will not possess a unique solution. Rather, there will be a continuum of points satisfying the equations. If (q_1, \dots, q_J) is one such solution, then $(q_1 + dq_1, \dots, q_J + dq_J)$ is also a solution provided the differentials, (dq_1, \dots, dq_J) , satisfy the relations

$$du_n = \sum_{j=1}^J (\partial u_n / \partial q_j) dq_j = 0 \quad (n = 1, \dots, N < J). \quad (9)$$

It is required that a solution point be chosen at which a given function, $\Phi(q_1, \dots, q_J)$, remains stationary with respect to differential variations consistent with Eqs. (9). At such a point, then,

$$d\Phi = \sum_{j=1}^J (\partial \Phi / \partial q_j) dq_j = 0, \quad (10)$$

where the differentials satisfy Eqs. (9).

At this juncture, the artifice of Lagrange multipliers may be introduced. If Eqs. (9) are satisfied by a set of differentials (which was the condition under which the extremum of Φ was defined above), then

$$\sum_{n=1}^N \lambda_n du_n = \sum_{n=1}^N \sum_{j=1}^J \lambda_n (\partial u_n / \partial q_j) dq_j = 0 \quad (11)$$

is also satisfied by that set of differentials, where the λ_n are arbitrary multipliers. Thus Eq. (10) will be unchanged by addition of the sum,

$$\sum_{n=1}^N \lambda_n du_n.$$

Performing this addition, Eq. (10) becomes

$$d\Phi = \sum_{j=1}^J \left[(\partial \Phi / \partial q_j) + \sum_{n=1}^N \lambda_n (\partial u_n / \partial q_j) \right] dq_j = 0. \quad (12)$$

Now Eq. (12) will be satisfied if

$$(\partial \Phi / \partial q_k) + \sum_{n=1}^N \lambda_n (\partial u_n / \partial q_k) = 0 \quad (k = 1, \dots, J). \quad (13)$$

Eqs. (8) and (13) together form a set of $N + J$ equations in the $N + J$ unknowns, λ_n and q_j , and are potentially solvable for these unknowns. The resultant set of values, q_j , will then satisfy the extremum condition, Eq. (10), in addition to satisfying the equations of constraint.

V. Application to the Present Problem

The present problem requires the minimization of the function, Φ , given by Eq. (3a) subject to the constraints represented by Eqs. (4a). Thus we set

$$u_n(q_1, \dots, q_J) = \sum_{j=1}^J b_{nj} q_j, \quad (14)$$

from which

$$(\partial u_n / \partial q_k) = b_{nk}. \quad (15)$$

Also, from Eq. (3a),

$$\left(\partial \Phi / \partial q_k \right) = 2 \left[\sum_{m=1}^M \sum_{j=1}^J v_m^2 a_{mk} a_{mj} q_j - \sum_{m=1}^M v_m^2 a_{mk} d_m \right]. \quad (16)$$

Eqs. (13) thus become

$$\sum_{m=1}^M \sum_{j=1}^J v_m^2 a_{mk} a_{mj} q_j + \sum_{n=1}^N b_{nk} \psi_n = \sum_{m=1}^M v_m^2 a_{mk} d_m \quad (k = 1, \dots, J), \quad (17)$$

where $\lambda_n/2$ has been replaced by ψ_n .

Eqs. (17) and (4a) form a set of $N + J$ linear equations which may be solved by standard techniques for the $N + J$ unknowns, q_j and ψ_n .

Before proceeding further, it should be verified that the extremum of Φ guaranteed by the solution of Eqs. (17) and (4a) is actually a minimum. If Φ is expanded in a Taylor series about the extremum point, the first order terms vanish by virtue of Eq. (10). Thus in the neighborhood of the extremum, is given by the second order terms of the Taylor series:

$$\Phi - \Phi_0 = \sum_{j=1}^J \sum_{k=1}^J \left(\partial^2 \Phi / \partial q_j \partial q_k \right) \Delta q_j \Delta q_k, \quad (18)$$

where Φ_0 denotes the extremum value. It is assumed that the variations, Δq_j and Δq_k , are consistent with Eqs. (4a).

From Eq. (16),

$$\partial^2 \Phi / \partial q_j \partial q_k = \sum_{m=1}^M v_m^2 a_{mk} a_{mj}, \quad (19)$$

so that Eq. (18) becomes

$$\Phi - \Phi_0 = \sum_{j=1}^J \sum_{k=1}^J \sum_{m=1}^M v_m^2 a_{mk} a_{mj} \Delta q_j \Delta q_k = \sum_{m=1}^M \left[\sum_{j=1}^J v_m a_{mj} \Delta q_j \right]^2. \quad (20)$$

Thus,

$$\Phi - \Phi_0 \geq 0, \quad (21)$$

which shows that Φ_0 is a minimum.

If the equality in Eq. (21) holds for any allowed variations, then Φ is said to be semi-definite and the minimum is not uniquely defined. In this case, Eqs. (17) and (4a) do not possess a unique solution. The ambiguity in the solution may be removed, however, by adding to the sum

$$S = \sum_{j=1}^J (c_j q_j)^2, \quad (22)$$

where the c_j are weight factors. In addition to removing the ambiguity in the solution, the inclusion of the sum, S , affords a measure of control over the influence of the different variables on the solution. A large value for the weight factor c_k , for example, might be expected to yield a relatively small solution value, q_k .

If S is to be included in Φ , Eqs. (17) must be modified by adding the term

$$\frac{1}{2} \partial S / \partial q_k = c_k^2 q_k. \quad (23)$$

Eqs. (17) are thus replaced by

$$\sum_{n=1}^M \sum_{j=1}^J v_n^2 a_{nk} a_{nj} q_j + c_k^2 q_k + \sum_{n=1}^M b_{nk} \mathcal{D}_n = \sum_{n=1}^M v_n^2 a_{nk} d_n \quad (k = 1, \dots, J). \quad (24)$$

VI. Matrix Form of the Solution

The manipulation of Eqs. (24) and (4a) may be greatly facilitated by recasting the equations in matrix form. The necessary matrices are the following:

Appendix B

9.

the following:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1J} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MJ} \end{pmatrix}, \quad (25)$$

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1J} \\ \vdots & & \vdots \\ b_{N1} & \cdots & b_{NJ} \end{pmatrix}, \quad (26)$$

$$C = \begin{pmatrix} c_1^2 & \bigcirc \\ \vdots & \vdots \\ \bigcirc & c_J^2 \end{pmatrix}, \quad (27)$$

$$W = \begin{pmatrix} w_1 & \bigcirc \\ \vdots & \vdots \\ \bigcirc & w_M \end{pmatrix}, \quad (28)$$

$$d = \begin{pmatrix} d_1 \\ \vdots \\ d_M \end{pmatrix}, \quad (29)$$

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}, \quad (30)$$

10.

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_J \end{pmatrix}, \quad (31)$$

and

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}. \quad (32)$$

The matrix A and column matrix d will always appear multiplied by the matrix of weight factors W . For convenience these products are defined to be

$$M = WA \quad (33)$$

and

$$f = Wd. \quad (34)$$

Also, in order to conserve space, a matrix G and a column matrix g are defined by

$$G = M^t M + C \quad (35)$$

and

$$g = M^t f, \quad (36)$$

where the superscript, t , denotes the transpose.

With these definitions, Eqs. (24) and (4a) become, respectively,

$$Gq + B^t v = g \quad (37)$$

and

$$Bq = e. \quad (38)$$

While the solution of Eqs. (37) and (38) may be obtained through the inversion of a single matrix having dimensions $(NxJ) \times (NxJ)$, it is perhaps better for reasons of computational accuracy to arrive at the solution by inverting two smaller matrices. From Eq. (37),

$$q = G^{-1}(g - B^t v). \quad (39)$$

Substituting this result in Eq. (38) and solving for v ,

$$v = E^{-1}(BG^{-1}g - e), \quad (40)$$

where

$$E = BG^{-1}B^t. \quad (41)$$

Once v has been determined, it may be substituted in Eq. (39) to determine q . The matrices, G and E , which required inversion have dimensions $J \times J$ and $N \times N$, respectively. Both of these matrices are symmetric and hence require fewer operations for their inversion than do arbitrary matrices.

Since the solution of Eqs. (37) and (38) is based on the linear approximations expressed by Eqs. (5) and (6), it will not in general satisfy Eqs. (4) and yield a true minimum of Φ . However, if a solution does in fact exist for Eqs. (4), then Eqs. (37) and (38) may be applied iteratively to arrive at the required solution.

It is important that the changes in the variables produced at a given iteration be kept within reasonable limits so that the linear approximations retain some degree of validity. Otherwise, a system configuration may be generated which is so far removed from the final solution that the chance of convergence is reduced to the vanishing point. If the e_n have been reduced to zero, the weight factors, c_j , may be used to limit the changes produced at a given iteration. If the e_n are not all zero, however, it is impossible to confine the changes in all of the variables within arbitrary limits by use of

the weights, c_j . In such cases one may take

$$p_j = p_j^i + kq_j \quad (j = 1, \dots, J), \quad (42)$$

where k is assigned a small enough value to prevent any of the p_j from exceeding specified limits.

VII. Choice of Weight Factors

The weight factors, w_m and c_j , play an important role in determining both the nature of the solution obtained and the speed with which the method converges to the solution. It has already been indicated that the factors c_j control the effectiveness of the different variables. As a factor c_k is made larger, the corresponding variable, q_k , is forced toward a smaller solution value. This control should be of particular value in preventing variables from straying far beyond physically realizable limits. The factors w_m determine the emphasis placed upon the corresponding terms of Φ . A large value for a particular factor, w_k , will lead to a solution for which the k -th term of Φ is likely to be reduced to a greater extent than if a lesser value of w_k were used. The overall balance between the values of the factors w_m and the factors c_j determines whether the major emphasis is placed on the minimization of the quadratic approximation to Φ or on keeping the changes in the system variables relatively small from iteration to iteration.

Even when all the weight factors are equal, however, there may be an artificial weighting which arises from the fact that both variables and performance functions may be a variety of types having widely differing ranges. For example, a variation of 10^{-10} in the value of a high order aspheric deformation coefficient might produce changes in a set of performance functions of the same order of magnitude as those produced by a variation of 10 in the value of a lens element thickness. Thus, with equal weights, c_j , for these two

variables, a solution would be obtained in which virtually all of the "work" was done by the aspheric coefficient. Similarly, if one term of Φ had a larger value and larger derivatives than any of the other terms, the greatest emphasis would be placed on the reduction of that term.

It is of primary importance that the effects of artificial weighting of the variables be counteracted. Because of the large differences among the variables which may be encountered in practice, it is possible for certain variables to be rendered almost totally ineffective by artificial weighting. If significant changes in these variables are necessary to the final solution, the method may fail to converge or, at best, will converge slowly.

A possible way in which to counteract the artificial weighting of the variables is to require that

$$(\partial^2 s / \partial q_j^2) = (\partial^2 \Psi / \partial q_j^2), \quad (43)$$

where

$$\Psi = \sum_{m=1}^M (e_m - s_m)^2 + \sum_{n=1}^N (h_n - t_n)^2. \quad (44)$$

This yields

$$c_j^2 = \sum_{m=1}^M a_{mj}^2 + \sum_{n=1}^N b_{nj}^2. \quad (45)$$

In order to retain the ability to control the weighting of variables, c_j^2 may be replaced by

$$c_j^2 = \bar{c}_j^2 \left(\sum_{m=1}^M a_{mj}^2 + \sum_{n=1}^N b_{nj}^2 \right). \quad (46)$$

Equal values for the new weight factors, \bar{c}_j , will have the effect of making all of the variables about equally effective.

A numerical problem remains in that the elements of the matrices A and B may exhibit an extremely wide range of values so that round-off errors become troublesome. This problem may be alleviated by transforming to a new set of variables defined by

$$\bar{q}_j = \alpha_j q_j \quad (47)$$

where

$$\alpha_j = \left(\sum_{m=1}^M a_{mj}^2 + \sum_{n=1}^N b_{nj}^2 \right)^{1/2}. \quad (48)$$

Eqs. (37) and (38) then become

$$\bar{U} \bar{q} + \bar{B}^t \bar{v} = \bar{g} \quad (49)$$

and

$$\bar{B} \bar{q} = e, \quad (50)$$

where

$$\bar{U} = \alpha^{-1} M^t \alpha^{-1} + \bar{C} \quad (51)$$

and

$$\bar{B} = B \alpha^{-1}. \quad (52)$$

The matrices α^{-1} and \bar{C} are the diagonal matrices:

$$\alpha^{-1} = \begin{pmatrix} \alpha_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \alpha_J^{-1} \end{pmatrix} \quad (53)$$

and

$$\bar{C} = \begin{pmatrix} \bar{c}_1^2 & & 0 \\ & \ddots & \\ 0 & & \bar{c}_J^2 \end{pmatrix} \quad (54)$$

It is evident from the above that the effect of artificial weighting of the variables disappears with proper scaling of the variables.

VIII. Particular Cases of Interest

Several previously described methods of automatic lens correction appear within the framework of the present formulation. It should be instructive, therefore, to review them.

The Method of Least Squares used by Rosen and Eldert⁷, and by Meiron⁸, deals with the minimization of Φ in the absence of constraints. Thus B, C, and e become null matrices and Eq. (39) reduces to

$$q = G^{-1}g = (M^t M)^{-1} M^t f. \quad (55)$$

This method has been used with some success to reduce the magnitude of spot diagram ray deviations and to minimize third order aberrations. It is necessary that the number of variables be less than or equal to the number of terms of Φ . Otherwise the solution will be indeterminate. In some cases excessively large values of the q_j may be produced. The changes applied to the variables must then be limited in the manner indicated by Eq. (42).

Feder⁹ has pointed out that while the method may converge relatively quickly to the vicinity of a minimum, the accuracy of the method in that vicinity may be so impaired by nonlinearity as to prevent further development.

Hopkins and McCarthy^{10, 11} have described a procedure designed to yield explicit values for a set of system functions when the number of available variables exceeds or equals the number of functions. (In their application, the system functions were the third order monochromatic and first order chromatic aberrations.) The procedure follows from the present formulation by eliminating Φ and setting $c_j = 1$ for all j . Thus A and d become null matrices, G becomes the identity matrix, and Eq. (40) reduces to

$$\mathbf{v} = -\mathbf{E}^{-1}\mathbf{e} = -(\mathbf{B}\mathbf{B}^t)^{-1}\mathbf{e}. \quad (56)$$

Eq. (39) then yields

$$\mathbf{q} = -\mathbf{B}^t\mathbf{v} = \mathbf{B}^t(\mathbf{B}\mathbf{B}^t)^{-1}\mathbf{e}. \quad (57)$$

This method has been used for several years at the Institute of Optics and has proved to be a powerful aid in practical design¹². The condition that the sum of the squares of the q_j be minimized tends to confine successive changes in the system configuration within reasonable limits when the elements of \mathbf{e} are not excessively large nor the elements of \mathbf{B} excessively small. In cases for which large changes have been generated, recourse to the limiting procedure represented by Eq. (42) has usually proved successful in preventing the process from diverging.

Recently, Wynne^{13, 14} has described a method of "Successive Linear Approximation at Maximum Steps", which is an extension of the Method of Least Squares. Here one includes the sum S but sets all of the weight factors, c_j , equal. Thus

$$\mathbf{C} = c\mathbf{I}, \quad (58)$$

where \mathbf{I} is the identity matrix. Eq. (39) then becomes

$$\mathbf{q} = (\mathbf{M}^t\mathbf{M} + c\mathbf{I})^{-1}\mathbf{M}^t\mathbf{f}. \quad (59)$$

This method provides control over the magnitude of the q_j at each iteration through the choice of a value for c . As c is increased, the magnitude of \mathbf{q} (considered as a J dimensional vector) is decreased and the direction of \mathbf{q} is shifted toward the direction of the negative gradient of Φ at the point $\mathbf{q} = 0$, i.e., toward the direction of maximum decrease in Φ . The ability to restrict the q_j to a region of approximate validity of Eqs. (6), coupled with the removal of any indeterminacy in the solution, assures convergence. It is necessary to choose c carefully if the full potential of the method is to be

realized, however. Too large a value will overly restrict the changes in the variables so that the speed of convergence will be unnecessarily reduced. Too small a value may result in less than optimum convergence because of non-linearity.

IX. Conclusions

It is not anticipated that the method described here will be a panacea for all problems of lens design. It is expected, however, that the method will provide the lens designer with a particularly flexible means of exploiting the advantage of high speed automatic calculation offered by modern computing machinery. The power of the method will depend largely on the skill which the designer displays in controlling its use.

The choice of system evaluation functions should depend not only on the particular requirements of a given design problem but also on the stage of development of the design. At the outset it may be most expedient to work with simple evaluation functions, e.g., the third and fifth order aberrations, leaving the use of more definitive but computationally more complex evaluation functions to a later stage. If reasonable values cannot be obtained for the simple functions it will be necessary to make a major alteration in the system, either by adding aspherics or additional elements, or by selecting an entirely different initial configuration. In such circumstances the additional time required to compute values for more elaborate functions from the outset would be wasted.

It has been mentioned that the choice of weight factors for the terms of Φ determines the nature of the solution obtained. As a design progresses, the judgment of the designer in modifying the weight factors, so that emphasis is shifted from one characteristic of the system performance to another, will be influential in determining the quality of the final design.

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Footnotes

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Appendix C

INTRODUCTION TO THE GEOMETRIC OPTICAL FREQUENCY RESPONSE

I. Transition from the Wave Optical to the Geometric Optical Response

The wave optical intensity response is given by

$$\gamma_{\omega}(N_x, N_y) = K^{-1} \iint_{-\infty}^{\infty} A(X, Y) A^*(X - \lambda R N_x, Y - \lambda R N_y) dX dY, \quad (1)$$

where

$$A(X, Y) = |A(X, Y)| e^{\frac{2\pi i}{\lambda} \phi(X, Y)} \quad (2)$$

and

$$K = \iint_{-\infty}^{\infty} |A(X, Y)|^2 dX dY. \quad (3)$$

λ represents the wavelength of light in the image space; R , the radius of the Gaussian reference sphere; (N_x, N_y) , the spatial frequency components; (X, Y) , the coordinates of a general point on the reference sphere, referred to coordinate axes in the exit pupil plane.

$A(X, Y)$ is called the complex aperture transmission function, or simply the aperture function. It specifies the amplitude and phase variation, over the reference sphere, of the light originating at a monochromatic object source point. $|A(X, Y)|$ is usually assumed to have a constant value of unity within the physical limits defined by the exit pupil and zero value outside these limits. (This assumption is, of course, invalid for systems having apodizing apertures.)

$\phi(X, Y)$ is called the wave aberration. It is defined with respect to a constant phase wavefront in the neighborhood of the exit pupil. To be definite, we may take the wavefront which passes through the origin, $(0, 0)$,

at the exit pupil. $\phi(X,Y)$ is the distance from a point on the wavefront to the point (X,Y) on the reference sphere, measured along the normal to the wavefront (i.e., along the ray which intersects the reference sphere at (X,Y)).

Our purpose in this section is to deduce the form of the response function in the short wavelength limit, $\lambda \rightarrow 0$. This is the geometric optical approximation.

For convenience, we first write

$$A(X,Y) A^*(X - \lambda R N_x, Y - \lambda R N_y) = r e^{i\psi} \quad (4)$$

where, from (2),

$$r = |A(X,Y)| |A(X - \lambda R N_x, Y - \lambda R N_y)| \quad (5)$$

and

$$\psi = \frac{2\pi}{\lambda} [\phi(X,Y) - \phi(X - \lambda R N_x, Y - \lambda R N_y)] \quad (6)$$

We see immediately that

$$\lim_{\lambda \rightarrow 0} r = |A(X,Y)|^2, \quad (7)$$

so our attention is directed to the function ψ .

The most obvious method of attack is to expand ψ as a power series in λ about $\lambda = 0$. To do this we consider $\phi(\alpha, \beta)$ where

$$\left. \begin{aligned} \alpha &= X - \lambda R N_x \\ \beta &= Y - \lambda R N_y \end{aligned} \right\} \quad (8)$$

The Taylor expansion of ϕ in the variable λ is

$$\phi = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\frac{\partial^n \phi}{\partial \lambda^n} \right)_{\lambda=0} \quad (9)$$

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Now $\partial/\partial\lambda$ may be thought of as an operator which is applied n times in the n th term of the series. We know that

$$\begin{aligned}\frac{\partial\phi}{\partial\lambda} &= \frac{\partial\phi}{\partial\alpha} \frac{\partial\alpha}{\partial\lambda} + \frac{\partial\phi}{\partial\beta} \frac{\partial\beta}{\partial\lambda} \\ &= -R \left(N_x \frac{\partial\phi}{\partial\alpha} + N_y \frac{\partial\phi}{\partial\beta} \right),\end{aligned}$$

so that the operator $\partial/\partial\lambda$ is

$$\frac{\partial}{\partial\lambda} = -R \left(N_x \frac{\partial}{\partial\alpha} + N_y \frac{\partial}{\partial\beta} \right). \quad (10)$$

Hence we may write the series (9) as

$$\begin{aligned}\phi &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (-R)^n \left(N_x \frac{\partial}{\partial\alpha} + N_y \frac{\partial}{\partial\beta} \right)^n \phi(\alpha, \beta)_{\lambda=0} \\ &= \phi(x, y) - \lambda R \left(N_x \frac{\partial\phi(x, y)}{\partial x} + N_y \frac{\partial\phi(x, y)}{\partial y} \right) + \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} (-R)^n \left(N_x \frac{\partial}{\partial x} + N_y \frac{\partial}{\partial y} \right)^n \phi(x, y)\end{aligned} \quad (11)$$

In the preceding equation, we have made use of the fact that $(\alpha, \beta) = (x, y)$ when $\lambda = 0$ so that

$$\begin{aligned}\left(\frac{\partial\zeta(\alpha, \beta)}{\partial\alpha} \right)_{\lambda=0} &= \left(\frac{\partial\zeta(\alpha, \beta)}{\partial\alpha} \right)_{(\alpha, \beta) = (x, y)} = \frac{\partial\zeta(x, y)}{\partial x} \\ \left(\frac{\partial\zeta(\alpha, \beta)}{\partial\beta} \right)_{\lambda=0} &= \left(\frac{\partial\zeta(\alpha, \beta)}{\partial\beta} \right)_{(\alpha, \beta) = (x, y)} = \frac{\partial\zeta(x, y)}{\partial y}\end{aligned}$$

where $\zeta(\alpha, \beta)$ is any function of α and β .

Substituting (11) in (6):

$$\psi = 2\pi R \left(N_x \frac{\partial \phi}{\partial X} + N_y \frac{\partial \phi}{\partial Y} \right) - 2\pi \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{n!} (-R)^n \left(N_x \frac{\partial \phi}{\partial X} + N_y \frac{\partial \phi}{\partial Y} \right)^n \phi. \quad (12)$$

$$\text{Thus, } \lim_{\lambda \rightarrow 0} \psi = 2\pi R \left(N_x \frac{\partial \phi}{\partial X} + N_y \frac{\partial \phi}{\partial Y} \right). \quad (13)$$

It is assumed that none of the derivatives become infinite. There are unusual cases in which this assumption is violated (e.g., when a wave normal is tangent to the reference sphere), but the result (13) remains generally valid.

We are now able to write the geometric optical response using the results (13) and (7):

$$\tau_g(N_x, N_y) = \lim_{\lambda \rightarrow 0} \tau_{\text{sc}}(N_x, N_y) = K^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(X, Y)|^2 e^{2\pi R i \left(N_x \frac{\partial \phi}{\partial X} + N_y \frac{\partial \phi}{\partial Y} \right)} dX dY. \quad (14)$$

If we take $|A(X, Y)| = \begin{cases} 1, & (X, Y) \text{ within pupil area} \\ 0, & (X, Y) \text{ outside pupil area} \end{cases}$

then the normalizing factor, K , is simply the area, a , of the pupil, and the infinite limits in (14) may be replaced by the limits defined by the pupil. Thus

$$\tau_g(N_x, N_y) = a^{-1} \int \int_{\text{pupil}} e^{2\pi R i \left(N_x \frac{\partial \phi}{\partial X} + N_y \frac{\partial \phi}{\partial Y} \right)} dX dY.$$

(15)

II. Representation in Terms of Image Plane Ray Coordinates

Our purpose in this section is to relate the wave aberration, ϕ , to the image plane coordinates of the rays associated with the given wavefront. The result will be used in Eq. (15) to obtain an expression for the geometric optical response which depends only upon the image plane ray coordinates. The advantage of this representation comes from the fact that it is an easier task to calculate accurate ray coordinates than to calculate wave aberrations.

Fig. 1 shows the exit pupil plane, (X,Y) , the Gaussian reference sphere, S , the given wavefront, W , and the image plane, (X_k, Y_k) . The Gaussian reference sphere is assumed to pass through the origin at the exit pupil and to have its center of curvature at the Gaussian image point, (\bar{X}_k, \bar{Y}_k) , in the image plane. The image plane is assumed to be separated from the exit pupil by a distance D .

Now the distance from a point (X,Y,Z) on the wavefront to the Gaussian image point is

$$R' = \left[(X - \bar{X}_k)^2 + (Y - \bar{Y}_k)^2 + (Z - D)^2 \right]^{\frac{1}{2}}. \quad (16)$$

If we define Δ to be the distance from the point (X,Y,Z) on the wavefront to the reference sphere, measured along the normal to the reference sphere, we may write

$$R' = R + \Delta \quad (17)$$

where R is the radius of the reference sphere and is given by

$$R = \left[\bar{X}_k^2 + \bar{Y}_k^2 + D^2 \right]^{\frac{1}{2}} \quad (18)$$

From (16)-(18), we find

$$2R\Delta + \Delta^2 = X^2 + Y^2 + Z^2 - 2(X\bar{X}_k + Y\bar{Y}_k + ZD). \quad (19)$$

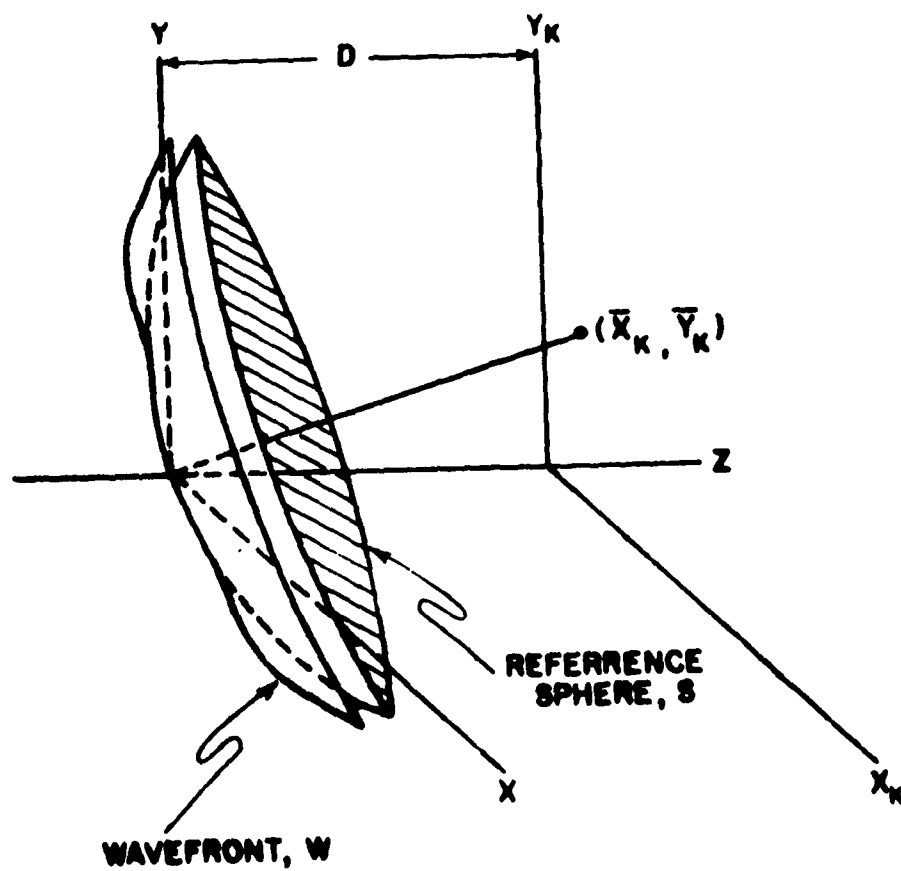


FIGURE 1. WAVEFRONT ASSOCIATED WITH POINT OBJECT, AS IT MIGHT APPEAR AT THE EXIT PUPIL.

Now typically,

$$\frac{\Delta}{R} \sim 10^{-5}, \quad (20)$$

so to a very good approximation we may drop the Δ^2 term in (19). The result may be taken as the equation of the wavefront and it will be convenient to write it in the form

$$F(X, Y, Z) = X\bar{X}_k + Y\bar{Y}_k + ZD - \frac{1}{2}(X^2 + Y^2 + Z^2) + R\Delta = 0. \quad (21)$$

Letting (X_k, Y_k) represent the image plane coordinates of a ray from (X, Y, Z) on the wavefront, the symmetric equations for the ray are

$$\frac{X_k - X}{\frac{\partial F}{\partial X}} = \frac{Y_k - Y}{\frac{\partial F}{\partial Y}} = \frac{D - Z}{\frac{\partial F}{\partial Z}} \quad (22)$$

Evaluating the derivatives, we find

$$\frac{X_k - X}{\bar{X}_k - X + R\frac{\partial \Delta}{\partial X}} = \frac{Y_k - Y}{\bar{Y}_k - Y + R\frac{\partial \Delta}{\partial Y}} = 1 \quad (23)$$

(Δ is a function of X and Y only; once it is given, the Z coordinate on the wavefront is determined by (21)). From (23) we immediately obtain the relations

$$\left. \begin{aligned} \delta X_k &\equiv X_k - \bar{X}_k = R\frac{\partial \Delta}{\partial X} \\ \delta Y_k &\equiv Y_k - \bar{Y}_k = R\frac{\partial \Delta}{\partial Y} \end{aligned} \right\} \quad (24)$$

To a good approximation we may replace Δ by ϕ , the wave aberration introduced in section I. The order of magnitude relation associated with

this approximation is

$$|\Delta - \delta| \sim \Delta^2 \quad (25)$$

Thus

$$\left. \begin{aligned} \delta X_k &\approx R \frac{\delta \phi}{\delta X} \\ \delta Y_k &\approx R \frac{\delta \phi}{\delta Y} \end{aligned} \right\} \quad (26)$$

These relations may be substituted in (15) to obtain the required expression for the response:

$$\tau_j(N_x, N_y) = a^{-1} \int \int_{\text{exit pupil}} e^{2\pi i (N_x \delta X_k + N_y \delta Y_k)} dX dY \quad (27)$$

One of the most direct ways of evaluating the above integral is to sum up spot diagram data. One traces a large number of rays through the optical system, arranging them such that they are uniformly distributed over the entrance pupil (and hence nearly uniformly distributed over the exit pupil). The response integral (27) is then replaced by the summation formula

$$\begin{aligned} \tau_j(N_x, N_y) &\approx \frac{1}{T} \sum_{j=1}^T \cos 2\pi [N_x (\delta X_k)_j + N_y (\delta Y_k)_j] \\ &\quad + \frac{i}{T} \sum_{j=1}^T \sin 2\pi [N_x (\delta X_k)_j + N_y (\delta Y_k)_j] \end{aligned} \quad (28)$$

where the subscript j identifies the j th ray, and T represents the total number of rays.

III.

Averaged Circular Response

Suppose we are concerned with imaging a sinusoidal target having spatial frequency, N , oriented at an angle θ with respect to the X axis in the object plane (see Fig. 2). The spatial frequency components then are

$$\left. \begin{aligned} N_x &= N \sin \theta \\ N_y &= N \cos \theta \end{aligned} \right\} \quad (29)$$

If we transform the ray deviations ($\delta x_k, \delta y_k$) to polar form,

$$\left. \begin{aligned} \delta x_k &= \delta \rho_k \cos \theta_k \\ \delta y_k &= \delta \rho_k \sin \theta_k \end{aligned} \right\} \quad (30)$$

we may write (27) as follows:

$$\begin{aligned} T_g(N, \theta) &= a^{-1} \int \int_{\text{exit pupil}} e^{2\pi i N \delta \rho_k (\sin \theta \cos \theta_k + \cos \theta \sin \theta_k)} dX dY \\ &= a^{-1} \int \int_{\text{exit pupil}} e^{2\pi i N \delta \rho_k \sin(\theta + \theta_k)} dX dY \end{aligned} \quad (31)$$

The average of (31) over all target orientations, θ , constitutes a useful criterion of system performance. We shall denote this average by $\bar{T}(N)$ and call it the averaged circular response. Using (31),

$$\begin{aligned} \bar{T}(N) &= (2\pi)^{-1} \int_0^{2\pi} T_g(N, \theta) d\theta = (2\pi a)^{-1} \int_0^{2\pi} \int \int_{\text{exit pupil}} e^{2\pi i N \delta \rho_k \sin(\theta + \theta_k)} dX dY d\theta \\ &= (2\pi a)^{-1} \int \int_{\text{exit pupil}} \int_0^{2\pi} e^{2\pi i N \delta \rho_k \sin \gamma} d\gamma dX dY \\ &= \frac{2}{\pi a} \int \int_{\text{exit pupil}} \int_0^{\pi/2} \cos(2\pi N \delta \rho_k \sin \gamma) d\gamma dX dY \end{aligned} \quad (32)$$

Now
$$\frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin Y) dY = J_0(x) \quad (33)$$

where $J_0(x)$ is the zero order Bessel function. Our response is thus

$$\bar{T}(N) = \frac{1}{\pi} \iint_{\text{exit pupil}} J_0(2\pi N \rho_k) dX dY \quad (34)$$

If spot diagram data is available, we may use the summation formula

$$\bar{T}(N) \approx \frac{1}{T} \sum_{j=1}^T J_0[2\pi N(\rho_k)_j] \quad (35)$$

where T is the number of rays and $(\rho_k)_j$ is the distance from the Gaussian image point to the point of intersection of the j th ray with the image plane.

It is common practice to present spot diagram data in the form of radial energy distribution curves. If we were to center a sufficiently small circular aperture at the Gaussian image point, a certain number of rays would fall outside and be blocked. If the radius of the aperture were allowed to increase, we would find more and more rays passing through until eventually all of the rays were admitted. The radial energy distribution curve is a plot of the fraction of the total number of rays passing through such an aperture as a function of the aperture radius. Each ray represents the light energy passing through an elemental area of the aperture so that a fraction of the total number of rays may be interpreted as a fraction of the total light energy passing through the optical system.

From a radial energy distribution curve we may select a set of points such as the following:

$$\begin{array}{rcl}
 E_0 = 0, & 0 \\
 E_1 = \frac{n_1}{T}, & (\delta \rho_k)_1 \\
 \vdots & \vdots \\
 E_i = \frac{n_i}{T}, & (\delta \rho_k)_i \\
 \vdots & \vdots \\
 E_M = 1, & (\delta \rho_k)_M
 \end{array}$$

where n_i is the number of rays passing through an aperture of radius $(\delta \rho_k)_i$ centered at the Gaussian image point, and T is the total number of rays in the spot diagram. The points are chosen such that $n_{i+1} > n_i$ (see Fig. 3).

If the points are not too widely separated, we may say to a fair approximation that the spot diagram consists of

$$\begin{array}{rcl}
 n_1 & \text{ray points having radius} & \frac{(\delta \rho_k)_1}{2} \\
 n_2 - n_1 & " & \frac{(\delta \rho_k)_2 + (\delta \rho_k)_1}{2} \\
 \vdots & \vdots & \vdots \\
 n_i - n_{i-1} & " & \frac{(\delta \rho_k)_i + (\delta \rho_k)_{i-1}}{2} \\
 \vdots & \vdots & \vdots \\
 n_M - n_{M-1} & " & \frac{(\delta \rho_k)_M + (\delta \rho_k)_{M-1}}{2}
 \end{array}$$

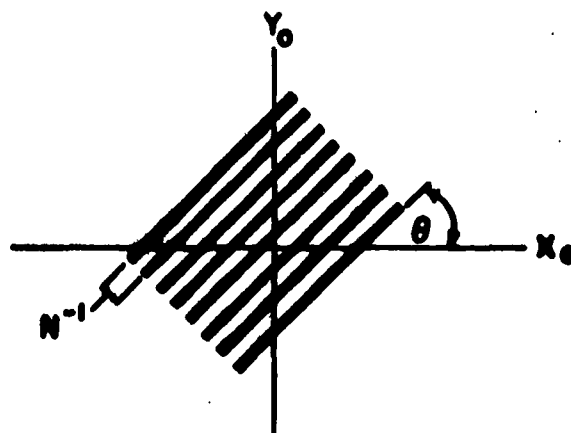


FIGURE 2

SINUSOIDAL TARGET IN THE OBJECT PLANE.

FREQUENCY COMPONENTS : $N_x = N \sin \theta$, $N_y = N \cos \theta$

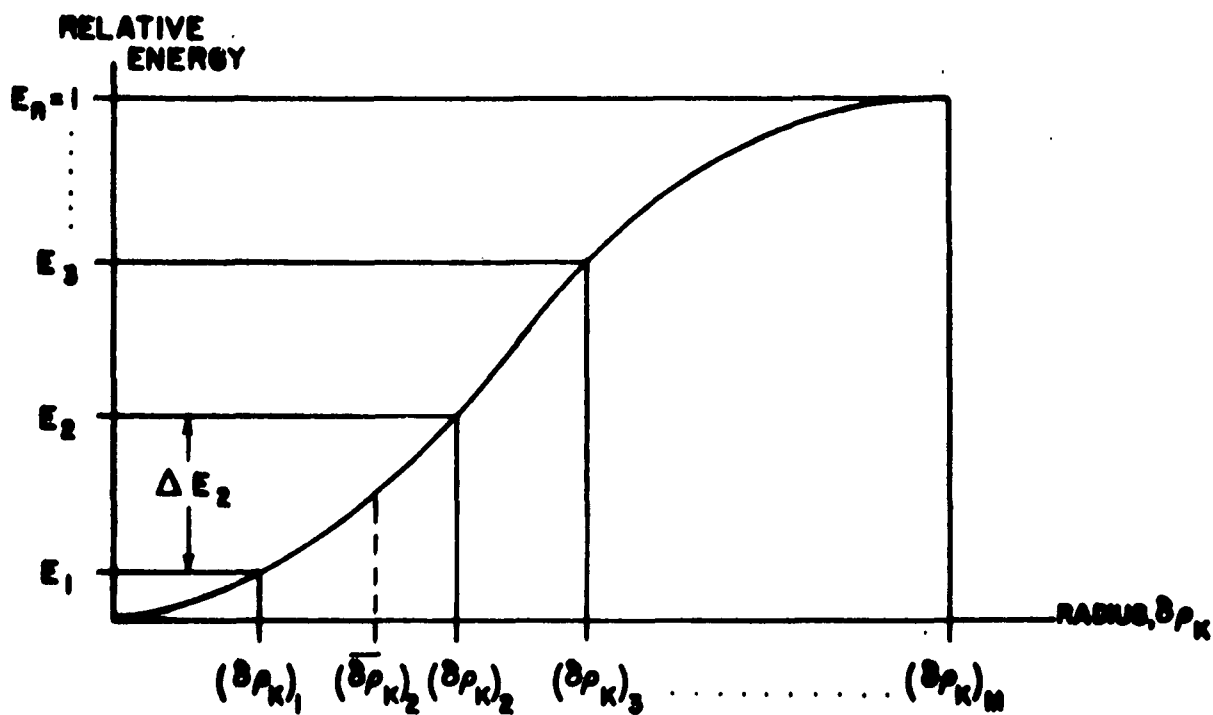


FIGURE 3

TYPICAL RADIAL ENERGY DISTRIBUTION CURVE SHOWING VALUES USED FOR APPROXIMATE AVERAGED CIRCULAR RESPONSE CALCULATIONS.

Appendix C

Thus, (35) may be written

$$\bar{P}(N) \approx \frac{1}{T} \sum_{i=1}^M \Delta n_i J_0 [2\pi N (\overline{\delta p_k})_i], \quad (36)$$

where

$$\Delta n_i = n_i - n_{i-1} \quad (37)$$

and

$$(\overline{\delta p_k})_i = \frac{(\delta p_k)_i + (\delta p_k)_{i-1}}{2}. \quad (38)$$

Noting that

$$\frac{\Delta n_i}{T} = E_i - E_{i-1} \quad (39)$$

we rewrite (36):

$$\bar{P}(N) = \sum_{i=1}^M \Delta E_i J_0 [2\pi N (\overline{\delta p_k})_i] \quad (40)$$

where

$$\Delta E_i = E_i - E_{i-1} \quad (41)$$

The formula (40) requires much less computing time than does (35). Typically, a value of 20 for M is sufficient to give acceptable accuracy for design work.

IV.

Series Expansion of the Averaged Circular Response --

Moments of the Spot Diagram Distribution

A typical term of the series (35) may be expanded to yield

$$J_0 [2\pi N (\delta p_k)_j] = \sum_{m=0}^{\infty} \frac{(-1)^m [\pi N (\delta p_k)_j]^{2m}}{(m!)^2} \quad (42)$$

Thus

$$\bar{T}(N) \cong \frac{1}{T} \sum_{j=1}^T \sum_{m=0}^{\infty} \frac{(-1)^m [\pi N(\delta p_k)_j]^{2m}}{(m!)^2} = \sum_{m=0}^{\infty} K_m \alpha_m N^{2m}, \quad (43)$$

where

$$K_m = \frac{(-\pi^2)^m}{(m!)^2} \quad (44)$$

and

$$\alpha_m = \frac{1}{T} \sum_{j=1}^T (\delta p_k)_j^{2m}. \quad (45)$$

The α_m may be recognized as the even moments of the radial spot diagram distribution. Recent applications of automatic computing techniques to the design of optical systems have been based on α_1 , as a measure of system performance. We see that α_1 determines the response in the low frequency region, and that minimization of α_1 for a given image will maximize the low frequency response. This gain, however, may be at the sacrifice of good high frequency response. The following example illustrates this point.

Suppose we have an image characterized by the equation

$$\delta p_k = c\rho - 0.1\rho^7 \quad (46)$$

where ρ is the radial exit pupil coordinate:

$$\rho = (x^2 + y^2)^{\frac{1}{2}}. \quad (47)$$

Appendix C

The first term of (46) represents defect of focus, while the second term represents undercorrected seventh order spherical aberration. We shall take the coefficient, c , of the first term of (46) as a variable which allows us to alter the image distribution. Variation of c corresponds to a movement of the image plane along the Z axis.

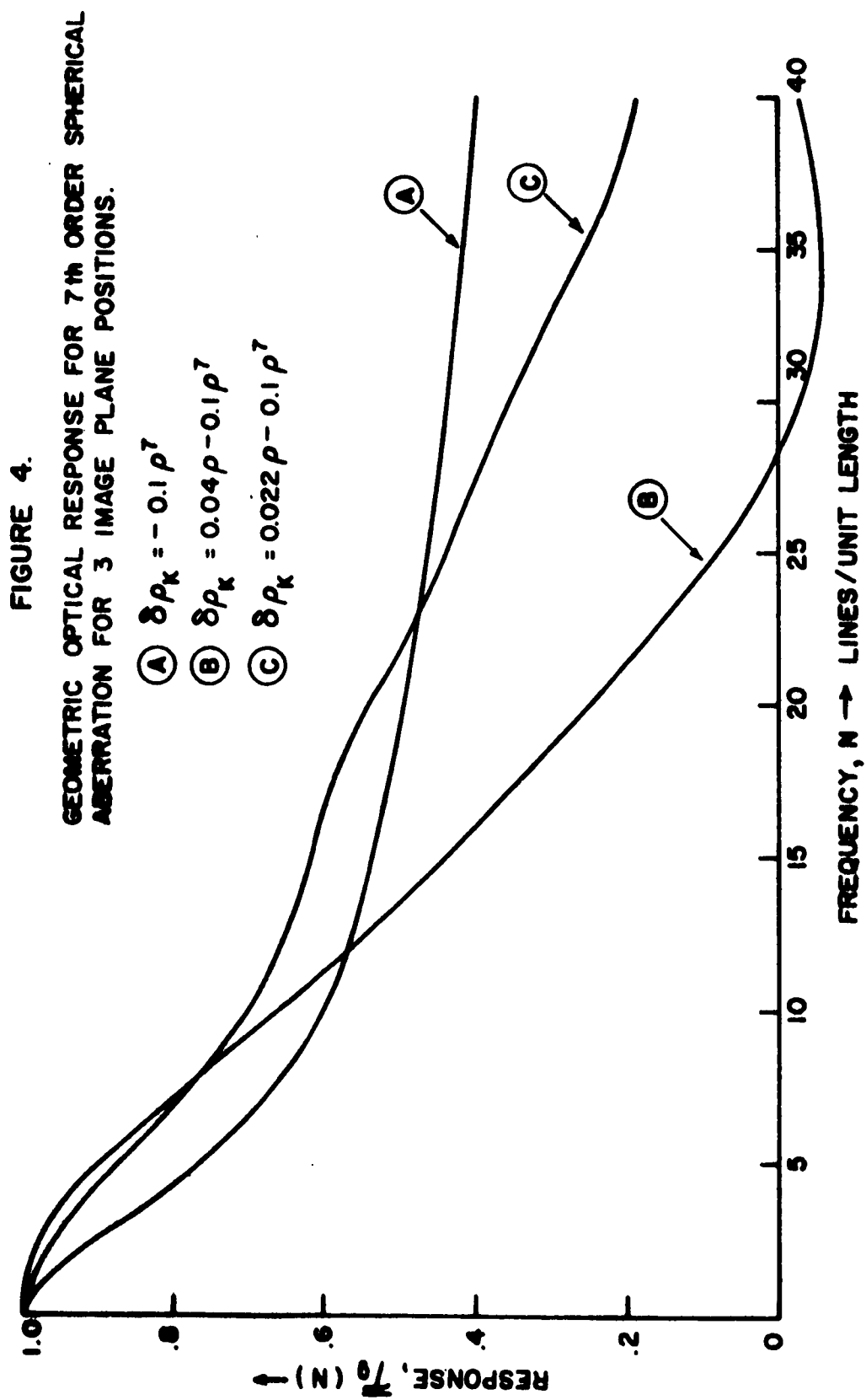
Fig. (4) shows the response $\tilde{T}(N)$ for several values of c , assuming an aperture defined by

$$\rho \leq 1. \quad (48)$$

The curve labelled A is obtained when $c = 0$. The response is seen to drop fairly rapidly in the low frequency region to a value of about 0.6 and then to level out, extending well into the higher frequency region without dropping below 0.4. Fine detail in the object will thus be preserved in the image, although at reduced contrast.

The curve labelled B is obtained when $c = .04$. This is the value of c for which α_1 is minimum. The response in the low frequency region has been considerably improved. In the higher frequency region, however, the response now drops quite rapidly to zero and becomes slightly negative, indicating "spurious resolution".

The curve labelled C corresponds to an intermediate image plane position, $c = .022$. The improvement over curve A extends well into the mid frequency region. Curve C is a decided improvement over curve B, the slight reduction in low frequency response being more than compensated for by the large elevation in the higher frequency response. (In special applications, of course, curve B may be preferred for specific reasons determined by the application.)



Curve C corresponds to the minimum of the function

$$= \frac{1}{T} \sum_{j=1}^T \frac{(\delta e_n)_j^2}{1 + (\delta e_n)_j^2 / \alpha_1}, \quad (49)$$

which shows some promise of being a more suitable "merit function" than α_1 for general purpose design.

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